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Wave forces on vertical axisymmetric bodies

By JARED L. BLACK
Chevron Oil Field Research Company, La Habra, California 90631

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An integral-equation formulation is used to calculate wave forces on bodies having a vertical axis of symmetry. The development enables one to calculate the forces without completely determining the local pressure field, thus offering a considerable reduction of computational effort. Numerical results are presented for a hemisphere at the water surface and vertical circular cylinders.

1. Introduction

The purpose of this paper is to present an integral-equation formulation for calculating the forces induced by water waves incident on a rigid body having a vertical axis of symmetry. This general technique was used by John (1950) in formulating a method for calculating wave forces on arbitrarily shaped bodies and several authors (Garrison & Chow 1972; Lebreton & Cormault 1969) have presented numerical results based on this formulation. These results demonstrate the generality of the method and its versatility with regard to body form; however, forces are obtained through the solution of a large matrix equation, a time-consuming operation.

In certain cases it is possible to reduce the numerical requirements by tailoring the problem formulation to a particular class of body shapes. Black, Mei & Bray (1971) studied wave forces on vertical cylindrical bodies defined by constant co-ordinate surfaces (see also Miles & Gilbert 1968; Garrett 1971) and obtained solutions efficiently through the application of Schwinger's variational technique.

Havelock (1955) used a spherical co-ordinate system in formulating the problem of a hemisphere heaving at the free surface of an infinitely deep fluid. The approximate solution, based on the four leading terms of an infinite series, agrees well with results presented herein. Milgram & Halkyard (1971) also studied the heaving-hemisphere problem and presented two solutions based on the integral-equation method. One was a straightforward application of John's formulation and required the solution of 64 simultaneous equations. By taking advantage of the problem symmetry, the second method required the solution of only eight simultaneous equations thus offering a considerable saving of numerical effort. This computational procedure could not be extended beyond the heaving problem because symmetry, a requirement of its application, was lost.

In this paper, we shall develop an integral-equation formulation that allows efficient calculation of wave forces on any smooth contoured body having a vertical axis of symmetry. The formulation is based on linear wave theory (Stoker 1957) and assumes finite water depth (see figure 1). The development is

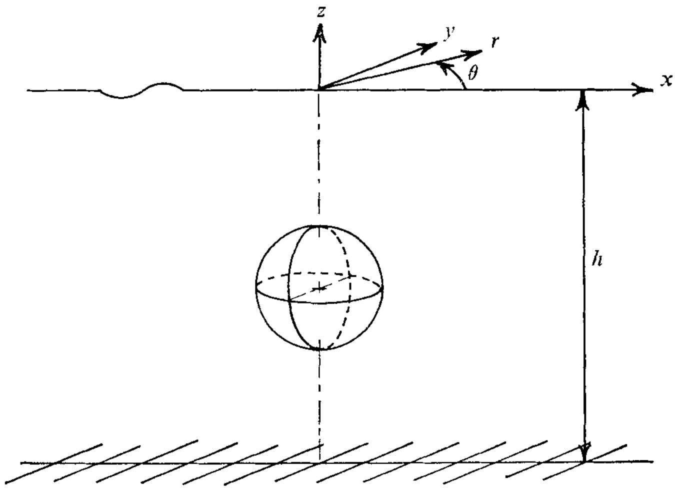

FIGURE 1. Definition sketch showing a submerged sphere.

summarized as follows. The incident and unknown scattered wave potentials and Green's function are written as expansions in cylindrical waves. By invoking Green's theorem, the scattered wave potential is expressed as the unknown in a Fredholm integral equation. Integrating over the angular dependence reduces the integral equation from a surface integral to an infinite number of line-integral equations; however, because the body has vertical symmetry, only two of the reduced equations contribute to the net force on the body. The determination of the force through the solution of the two reduced integral equations is an efficient operation relative to the effort required to solve a three-dimensional integral equation.

This formulation is of practical ocean engineering interest because offshore facilities, such as single-point mooring dolphins, subsurface storage tanks and buoyant components of floating platforms, often possess vertical symmetry.

2. Boundary-value problem

The governing equation for the velocity potential $\phi e^{i\omega t}$ and Green's function G, assuming small amplitude irrotational motion (Wehausen & Laitone 1960), is

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right)\begin{pmatrix}\phi\\ G\end{pmatrix}=\begin{pmatrix}0\\ \delta(r-\eta)\,\delta(\theta-\beta)\,\delta(z-\xi)\,r^{-1}\end{pmatrix}, \tag{2.1a}$$

where (r,θ,z) are the cylindrical co-ordinates of a point in the fluid domain, (η,β,ξ) are the co-ordinates of a source point and δ is the Dirac delta function. The free-surface condition is

$$(\partial/\partial z-\sigma)\,(\phi,G)=0, \quad z=0, \tag{2.1b}$$

where $\sigma=\omega^2/g$, and the bottom condition is

$$\partial(\phi,G)/\partial z=0, \quad z=-h. \tag{2.1c}$$

In addition to (2.1) the velocity potential must satisfy

$$\nabla\phi.\mathbf{n} = 0 \tag{2.2}$$

on the body to ensure zero flow normal to the body surface; $\mathbf{n}$ is the unit normal vector oriented out of the fluid.

The velocity potential is separated into incident and scattered components $\phi^i+\phi^s$ with the condition that the scattered potential represents outward-propagating waves. The incident wave is specified as a plane progressive wave of amplitude a_0 propagating in the positive-x direction. Its potential in cylindrical co-ordinates is (Miles & Gilbert 1968)

$$\phi^i = a_1 f_1(z) \sum_{m=0}^{\infty} \epsilon_m(-i)^m J_m(kr)\cos m\theta. \tag{2.3}$$

This expression satisfies (2.1) and is related to the incident wave amplitude through $a_0 = -(i\omega/g)\,a_1 f_1(0)$.

A Green's function, developed on the basis of a method outlined by Morse & Feshbach (1953, chap. 7), satisfying (2.1) is

$$G(r,\theta,z/\eta,\beta,\xi) = \sum_{m=0}^{\infty} \epsilon_m g^m(r,z/\eta,\xi)\cos m\theta\cos m\beta, \tag{2.4a}$$

where
$$g^m(r,z/\eta,\xi) = \frac{i}{4}f_1(z)f_1(\xi)\begin{pmatrix} H_m(kr)\,J_m(k\eta) \\ H_m(k\eta)\,J_m(kr)\end{pmatrix} - \frac{1}{2\pi}\sum_{n=2}^{\infty} f_n(z)f_n(\xi)\begin{pmatrix} K_m(k_n r)\,I_m(k_n\eta) \\ K_m(k_n\eta)\,I_m(k_n r)\end{pmatrix}. \tag{2.4b}$$

The upper terms in the brackets are used when $r > \eta$ and the lower terms when $\eta > r$. Terms appearing in (2.3) and (2.4) are defined as follows:

$$\epsilon_0 = 1, \quad \epsilon_m = 2, \quad m \geqslant 1,$$
$$f_n(z) = 2^{\frac{1}{2}}(h-\sigma^{-1}\sin^2 k_n h)^{-\frac{1}{2}}\cos k_n(z+h),$$

where the k_n are defined by

$$k_n \tan k_n h = -\sigma, \quad k_1 = ik.$$

The Hankel function $H_m(x)$ in (2.4) is of the second kind and satisfies the radiation condition; J_m, K_m and I_m are standard Bessel functions.

Equation (2.4) differs from the Green's function John used in that it is non-singular; the point source is represented by a discontinuity in the gradient of the function rather than a singularity. Thorne (1953) treats the general class of singular Green's functions, but since they are not completely separable, they do not serve a useful purpose in our formulation.

3. Scattered wave potential

The scattered wave potential is expressed in terms of the Green's function by invoking Green's theorem. By virtue of (2.1a), the volume integral in Green's theorem is readily integrated, leaving

$$\iint \{\phi^s\nabla G - G\nabla\phi^s\}.\mathbf{n}\,da = \left\{\begin{array}{ll}\phi^s, & \text{source in fluid,}\\ \tfrac{1}{2}\phi^s, & \text{source on boundary,}\\ 0, & \text{source outside boundary.}\end{array}\right\} \tag{3.1}$$

The right side of (3.1), by definition, accounts for the fact that only half the source flow enters the fluid domain when the source is on the boundary. The integration and differentiation is with respect to the (η, β, ξ) variables and the limits of integration include the body surface, the free surface, the bottom and a vertical cylindrical surface in the fluid at $\eta = \infty$. However, because of (2.1 b, c), the integrand vanishes on the free surface and the bottom; the integrand also vanishes on the surface at infinity because both the scattered wave potential and the Green's function represent outward-propagating waves. The integration is therefore performed only over the surface of the body.

Invoking (2.2) and allowing sources only on the surface of the body, (3.1) becomes

$$\tfrac{1}{2}\phi^s = \iint \phi^s \nabla G \,.\, \mathbf{n}\, da + \iint G \nabla \phi^i \,.\, \mathbf{n}\, da, \tag{3.2}$$

a Fredholm integral equation of the second kind. Defining

$$\phi^s = a_1 \sum_{m=0}^{\infty} \chi^m \cos m\theta, \quad \phi^i = a_1 \sum_{m=0}^{\infty} \psi^m \cos m\theta, \tag{3.3 a, b}$$

and substituting these along with (2.4 a) into (3.2), one obtains, upon integration over β,

$$\chi^m/4\pi = \int \chi^m \left(\frac{\partial g^m}{\partial \eta} n_\eta + \frac{\partial g^m}{\partial \xi} n_\xi\right) \eta\, dl + \int g^m \left(\frac{\partial \psi^m}{\partial \eta} n_\eta + \frac{\partial \psi^m}{\partial \xi} n_\xi\right) \eta\, dl, \quad m = 0, 1, 2, \ldots. \tag{3.4}$$

The unit normal vector $(n_\eta, 0, n_\xi)$ is oriented out of the fluid; the integration is carried out over the line defined by the intersection of the body with the plane $\theta = 0$. Note that because of symmetry $(n_\beta = 0)$ there is no mixing of angular modes.

Each integral equation is reduced to a linear system of equations by approximating the integration by the product of the integrand and the length of a straight-line segment of the contour; the result is

$$\sum_{j}^{N} (a_{ij}^m - \delta_{ij}/4\pi)\chi_j^m = b_i^m, \quad m = 0, 1, 2, \ldots, \tag{3.5}$$

where δ_{ij} is the Kronecker delta,

$$a_{ij}^m = \left\{\frac{\partial g_{ij}^m}{\partial \eta} n_{\eta j} + \frac{\partial g_{ij}^m}{\partial \xi} n_{\xi j}\right\} \eta_j \Delta l_j \tag{3.6}$$

and

$$b_i^m = -\sum_{j}^{N} g_{ij}^m \left\{\frac{\partial \psi_j^m}{\partial \eta} n_{\eta j} + \frac{\partial \psi_j^m}{\partial \xi} n_{\xi j}\right\} \eta_j \Delta l_j. \tag{3.7}$$

The subscripts on g and ψ indicate that the functions are evaluated at the ith and jth nodes, the subscripts relating to the node co-ordinates as follows: i for (r_i, z_i) and j for (η_j, ξ_j). The components of the normal vector are identified by the subscripts on n, the first subscript indicating the horizontal or vertical component and the second identifying the node. The length of the body-contour segment at the jth node is Δl_j.

4. Wave forces

The wave forces are obtained by integrating the pressure over the body. The pressure is, from Bernoulli's equation (Stoker 1957, p. 22),

$$p = -i\omega\rho\phi\, e^{i\omega t}, \quad \rho = \text{fluid density}.$$

At a point (r, θ, z) on the body, the pressure is

$$p_i = -i\omega\rho a_1 e^{i\omega t} \sum_{m=0}^{\infty} (\chi_i^m + \psi_i^m) \cos m\theta. \tag{4.1}$$

It can be easily shown that because of symmetry the vertical force depends only on the $m = 0$ angular mode and the horizontal force on the first mode, hence the forces are given by

$$F = C\, e^{i\omega t} = \frac{2\pi\rho g a_0\, e^{i\omega t}}{\epsilon_m f_1(0)} \sum_i^N (\chi_i^m + \psi_i^m)\, r_i n \Delta l_i, \tag{4.2}$$

where the horizontal force is obtained when $m = 1$ and $n = n_{ri}$ and the vertical force when $m = 0$ and $n = n_{zi}$. The moment about a y axis passing through $x = 0$, $z = CR$ is

$$M = \frac{\pi\rho g a_0 e^{i\omega t}}{f_1(0)} \sum_i^N (\chi_i^1 + \psi_i^1) \{(z_i - CR)\, r_i n_{ri} - r_i^2 n_{zi}\}\, \Delta l_i. \tag{4.3}$$

The detailed pressure distribution can be found, if desired, by solving for sufficient modes of (3.5) and substituting into (4.1).

5. Results

Forces are presented for two geometric shapes: a hemisphere at the surface and a vertical circular cylinder. The force is normalized according to

$$\mathscr{F} e^{i\alpha} = |C|\, e^{i\alpha}/(\rho g a_0 R^2),$$

where $$\alpha = \tan^{-1}\{\mathrm{Im}\, C/\mathrm{Re}\, C\}$$

and R is a characteristic dimension of the body. Subscripts x and z on $\mathscr{F}$ and α will indicate the horizontal and vertical components, respectively, of the force and its phase.

Figure 2 shows the normalized horizontal and vertical forces on a hemisphere at the water surface. Also shown is the vertical force obtained from Havelock's (1955) infinite-depth wave damping coefficient via Haskind's theorem (see Newman 1962). Good agreement with the infinite-depth result is exhibited at high and low frequencies and the influence of the finite water depth is apparent in the mid-frequency range for the case of $h/R = 2$; for $h/R = 4$, the finite- and infinite-depth solutions for the vertical force are essentially equal. The numerical result is based on a six-node specification of the body. Garrison & Chow (1972) and Milgram & Halkyard (1971) present numerical results based on John's formulation and use 132 and 64 nodes, respectively. The results of Milgram & Halkyard tend to diverge from the correct solution at high frequencies, perhaps because of insufficient node density.

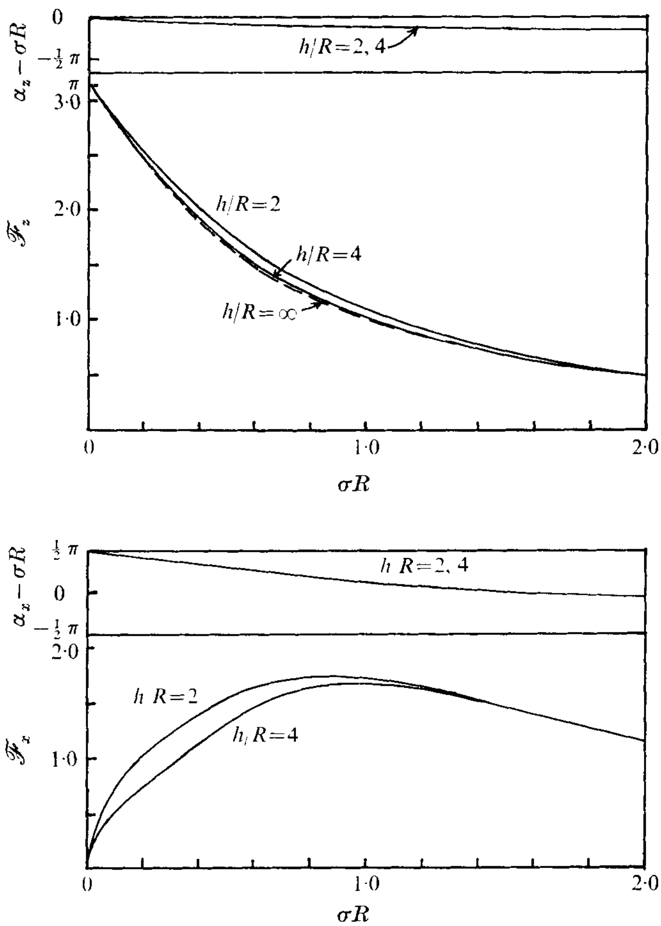

FIGURE 2. Forces on a sphere of radius R fixed with its origin at the free surface. The infinite-depth vertical force (dashed curve) is based on Havelock's heaving-hemisphere solution.

Forces on vertical cylinders are presented in figure 3. For a cylinder of height h and radius $R = \frac{1}{2}h$ the results agree with the MacCamy & Fuchs (1954) closed-form solution to within 1 % over the range of frequencies $0 < \sigma R < 2$. This accuracy was achieved with an eight-node specification of the cylinder. Garrison & Chow treated a similar cylinder problem and used 252 nodes to obtain equivalent accuracy. A comparison is also made with the variational solution presented by Black *et al.* (1971) for a cylinder radius $R = \frac{1}{2}h$ extending from the bottom up to one-half the water depth. The results agree to within 6 % for the horizontal force and 12 % for the vertical force. In this case one might expect only fair agreement because the sharp edge at the top of the cylinder is difficult to approximate with this numerical procedure. This body was specified with eight nodes.

The solutions presented here are approximate primarily because the boundary condition on the body is satisfied at a finite number of points. In principle, the

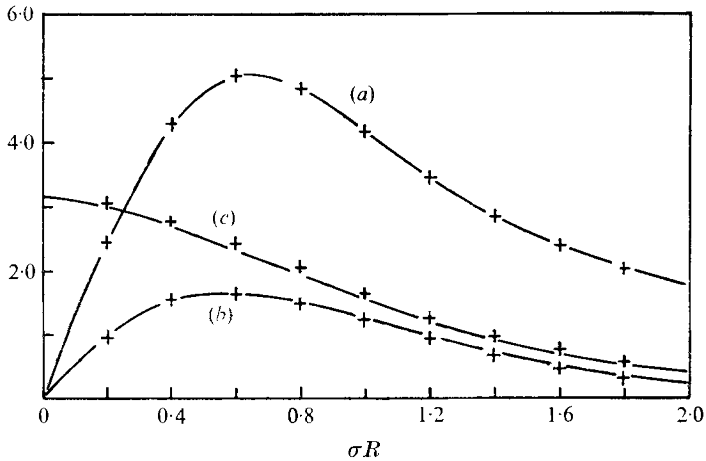

FIGURE 3. (*a*) Horizontal force on a vertical circular cylinder of radius $R = \frac{1}{2}h$ and height h: ———, closed-form solution; +, numerical result. (*b*) Horizontal force on a truncated vertical cylinder of radius $R = \frac{1}{2}h$ extending from the bottom to a height of $\frac{1}{2}h$: ———, variational solution; +, numerical result. (*c*) Vertical force on the truncated cylinder: ———, variational solution; +, numerical result.

solution accuracy can be improved by increasing the number of nodes used to describe the body; however, small node spacing adversely affects the rate of convergence of the series in (3.6) and (3.7). It was found by numerical trial that the series convergence rate was roughly $1/n$ for $i = j$ and the rate improved for the more widely separated nodes ($i \neq j$). This slow rate of convergence for $i = j$ seems to be a characteristic of series expansions of singular functions. In general, for $h/R < 10$, the series were sufficiently approximated by summing only 30 terms.

We have not considered the question of uniqueness and completeness of these solutions; it does seem intuitively reasonable, however, to assume that the conditions established by John (1950) apply to this formulation. John's uniqueness condition requires that a surface-piercing body intersect the free surface perpendicularly and that the body surface be a single-valued function within the intersection of the body contour and the free surface; the hemisphere satisfies this condition.

6. Conclusions

The primary advantage of this formulation is the computational efficiency it offers in determining the net body forces. Numerical trials indicate that the symmetric formulation generally produces solutions in one-sixtieth of the time required by our program based on John's formulation. The pressure distribution on the body is not a part of the wave-force solution but can be found by additional calculations. This solution method can be expanded to allow the determination of the hydrodynamic characteristics of a body (added mass and damping coefficient) without significantly increasing the computational requirements.

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